1. Symmetry

1.1. Central symmetry

The symmetry through point A is the transformation of the plane which sends point X into point X' such that A is the midpoint of segment XX'. The other names of such a transformation: the central symmetry with center A or just the symmetry with center A. If a figure turns into itself under the symmetry through point A, then A is called the center of symmetry of this figure.

Notice that the symmetry with center A is a particular case of two other transformations: it is the rotation through an angle of 180° with center A and also the homothety with center A and coefficient −1.

1.2. Symmetry through a line

The symmetry through a line l is a transformation of the plane which sends point X into point X' such that l is the midperpendicular to segment XX'. Such a transformation is also called the axial symmetry and l is called the axis of the symmetry. If a figure turns into itself under the symmetry through line l, then l is called the axis of symmetry of this figure.

The composition of two symmetries through axes is a parallel translation, if the axes are parallel, and a rotation, if they are not parallel. Axial symmetries are a sort of “bricks” all the other motions of the plane are constructed from: any motion is a composition of not more than three axial symmetries. Therefore, the composition of axial symmetries give much more powerful method for solving problems than compositions of central symmetries. Moreover, it is often convenient to decompose a rotation into a composition of two symmetries with one of the axes of symmetry being a line passing through the center of the rotation.

2. Rotations

To solve the problems it suffices to have the following idea on the notion of the rotation: a rotation with center O (or about the point O) through an angle of φ is the transformation of the plane which sends point X into point X' such that:

a) OX' = OX;

b) the angle from vector $\overrightarrow{OX}$ to vector $\overrightarrow{OX'}$ is equal to φ.

The problems solvable with the help of rotations can be divided into two big classes:

- problems which do not use the properties of compositions of rotations;
- problems which make use of these properties.

3. Homothety

3.1. General homothety

If the corresponding sides of two similar plygons are parallel, the two polygons are said to be similarly places or homothetic.

A geometric transformation of the plane is a function that sends every point on the plane to a point in the same plane. Here we will like to discuss one type of geometric transformations, called homothety, which can be used to solve quite a few geometry problems in some international math competitions. Two figures are called homothetic if the connectors of corresponding points are concurrent at a point which divides each connector in the same ratio.

A homothety is a transformation of the plane sending point X into point X' such that $\overrightarrow{OX'}$ = k × $\overrightarrow{OX}$, where point O and the number k are fixed. Point O is called the center of homothety and the number k the coefficient or ratio of homothety. So if |k| > 1, then the homothety is a magnification with center O. If |k| < 1, it is a reduction with center O. A homothety sends a figure to a similar figure.

Given a number k≠0 and point O of the plane, we call “homothety” of “center” O and “ratio” k the transformation which corresponds: a) to point O, itself, b) to every point X , O the point X' on the line OX, such that the following signed ratio relation holds:

$$\frac{OX'}{OX} = k$$

A direct consequence of the definition is, that for every point O the homothety of center O and ratio k = 1 is the identity transformation. Often, when the ratio is k < 0 we say that the transformation is an “antihomothety”. Its characteristic is that point O is between X and X'.

Theorem: The composition of two homotheties with center O and ratios k and w is a homothety of center O and ratio $k \times \lambda$.

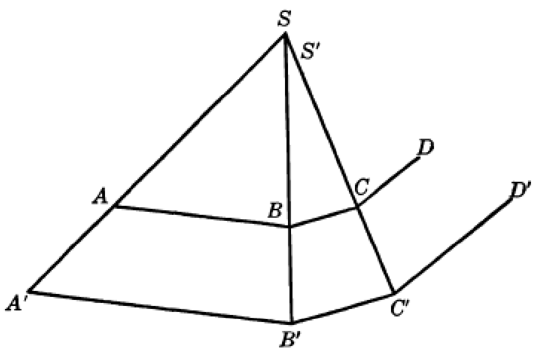
Proof: Obvious consequence of the definition. If f and g are the two homotheties with the same center O and ratios respectively k and λ, then, for every point X, points Y = f (X), Z = g(Y) and O will be four points on the same line and will satisfy,

$$\frac{OY}{OX} = k, \frac{OZ}{OY} = \lambda \ \rightarrow \ \frac{OZ}{OX} = \frac{OZ}{OY} \times \frac{OY}{OX} = k \times \lambda$$

The inverse transformation of a homothety f of center O and ratio k, is the homothety with the same center and ratio 1/k.

The homothety is a special transformation closely connected with the theorem of Thales and the “similarity of triangles”, i.e triangles which have equal corresponding angles ↔ triangles which have proportional corresponding sides. Homothetic triangles are particular cases of “similar triangles” and are the key to investigate properties of more general homothetic shapes.

Theorem: The lines joingin corresponding vertices of two homothetic polygons are concurrent (meet in a point).



Proof: Lets ABCD..., A'B'C'D'... be two homothetic polygons and S ϵ AA'×BB'. In order to provide the proposition it is necessary to show that a line joining the next pair C, C' of corresponding vertices will also pass through S. If the line CC' does not pass through S, let S' be the point where where it meets BB'. From the two pairs of similar triangles SAB and SA'B', S'BC and S'B'C' we have:

SB' : SB = A'B' : AB; S'B' : S'B = B'C' : BC.
But , by assumption: B'C' : BC = A'B' : AB
hence: SB' : SB = S'B' : S'B; SB' : (SB – SB') = S'B' : (S'B – S'B')

or SB' : BB' = S'B' : BB';
therefore S' coincides with S.

Theorem: For two triangles ABC and A'B'C', which have parallel corresponding sides, the lines AA', BB' and CC', which join the vertices with the corresponding equal angles, either pass through a common point and the triangles are homothetic, or are parallel and the triangles are congruent.

Proof: Let O be the intersection point of AA' and BB'. We will show that CC' also passes through point O. According to Thales, we have equal ratios:

$$\frac{AB}{A'B'} = \frac{OA}{O'A'} = \frac{OB}{O'B'}$$

Consider therefore on OC point C" with $\frac{OC}{OC''} = k$. The created triangle A'B'C" has sides proportional to those of ABC, therefore it is similar to it and consequently has the same angles. It follows, that A'B'C' and A'B'C" have A'B' in common and same angles at A' and B', therefore they coincide and C' = C", in other words, OC passes through C' too. This reasoning shows also that, if the two lines AA' and BB' do not intersect, that is if they are parallel, then the third line will also be necessarily parallel to them and ABB'A', BCC'B' and ACC'A' will be parallelograms, therefore the triangles will have corresponding sides equal.

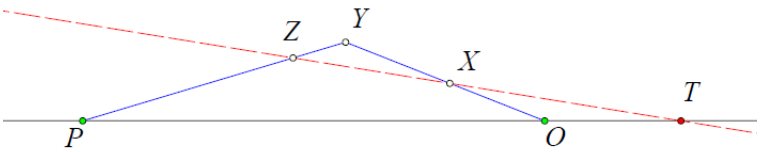
3.2. Rotational homothety

A rotational homothety is the composition of a homothety and a rotation with a common center.

3.3. Homotheties with different centers

Theorem: The composition of two homotheties f and g with different centers O and P and ratios respectively k and w, with $k \times \lambda \neq 1$, is a homothety with center T on the line OP and ratio equal to $k \times \lambda$.

Proof: The proof is an interesting application of the theorem of Menelaus. Let X be an arbitrary point and Y = f (X), Z = g(Y). This defines the triangle OYP and the points X, Z are contained in its sides OY and YP respectively. Let T be the intersection point of ZX with OP. Applying the theorem of Menelaus we have,



$$\frac{XO}{XY} \cdot \frac{ZY}{ZP} \cdot \frac{TP}{TO} = 1 \quad \Rightarrow \quad \frac{TP}{TO} = \frac{XY}{XO} \cdot \frac{ZP}{ZY}$$

$$XY = XO + OY \Rightarrow \frac{XY}{XO} = \frac{XO + OY}{XO} = 1 + \frac{OY}{XO} = 1 - \kappa,$$

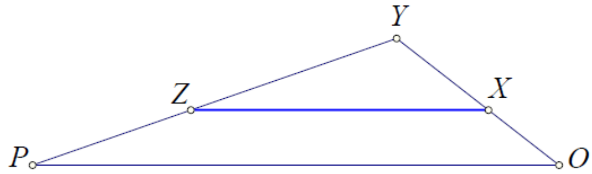
$$ZY = ZP + PY \Rightarrow \frac{ZY}{ZP} = \frac{ZP + PY}{ZP} = 1 + \frac{PY}{ZP} = 1 - \frac{1}{\lambda} \quad \Rightarrow$$

$$\frac{TP}{TO} = \frac{XY}{XO} \cdot \frac{ZP}{ZY} = (1 - \kappa) \cdot \left(\frac{1}{1 - \frac{1}{\lambda}} \right) = \frac{\lambda \cdot (1 - \kappa)}{\lambda - 1}.$$

The last formula shows, that the position of T on the line OP is fixed and independent of X. In addition, the ratio $\mu = \frac{TZ}{TX}$ is calculated, by applying the theorem of Menelaus to the triangle OXT, this time with PY as secant:

$$\begin{aligned}
\frac{PT}{PO}\cdot\frac{ZX}{ZT}\cdot\frac{YO}{YX} &= 1 && \Rightarrow \\
\frac{ZX}{ZT} &= \frac{YX}{YO}\cdot\frac{PO}{PT} && \Leftrightarrow \\
\frac{ZT+TX}{ZT} &= \frac{YO+OX}{YO}\cdot\frac{PT+TO}{PT} && \Leftrightarrow \\
1-\frac{1}{\mu} &= \left(1+\frac{OX}{YO}\right)\left(1+\frac{TO}{PT}\right) && \Leftrightarrow \\
1-\frac{1}{\mu} &= \left(1-\frac{1}{\kappa}\right)\left(1-\frac{\lambda-1}{\lambda(1-\kappa)}\right) && \Leftrightarrow \\
\mu &= \kappa\lambda\ .
\end{aligned}$$

Theorem: The composition of two homotheties f and g with different centers O and P respectively and ratios k and λ with k × λ = 1 is a translation by interval parallel to OP.



Proof: Let X be an arbitrary point and Y = f (X), Z = g(Y). This defines the triangle OYP and the points X, Z are contained in its sides OY and YP respectively. According to the hypothesis

$$\frac{YX}{YO} = \frac{YO+OX}{YO} = 1-\frac{1}{\kappa}, \qquad \frac{YZ}{YP} = \frac{YP+PZ}{YP} = 1-\frac{PZ}{YP} = 1-\lambda = 1-\frac{1}{\kappa}$$

The equality of the ratios shows, that the line segment XZ is parallel to OP. From the similarity of triangles YOP and Y XZ, follows that $XZ = (1-\lambda)OP$, therefor XZ has fixed length and direction.